Worked examples, grades 4 to 12

The number line, one step at a time

Every worked example from the interactive page, with every step written out: the line, the rule it uses, why it works, and the number line. Online version: https://number-line-steps.pages.dev

1 Rounding on the number line GRADE 4

Rounding means jumping to the closest “landmark” number. Put the number between its two landmarks and see which one it is nearer.

Example 1

Round 3,472 to the nearest hundred.

Step 1 of 5

Round 3,472 to the nearest hundred

> *Why?* The landmarks for “nearest hundred” are the multiples of 100: 3,300, 3,400, 3,500, 3,600, … Rounding means picking whichever landmark is closest.



Each small tick is 10.

Step 2 of 5

3,472 is between 3,400 and 3,500

FIND THE TWO LANDMARKS

> *Why?* Keep the digits up to the hundreds place and make the rest zeros: that gives the landmark just below, 3,400. One more hundred gives the landmark just above, 3,500.



Step 3 of 5

Halfway between them: 3,450

HALFWAY POINT

> *Why?* Halfway is the dividing line. Below halfway, the lower landmark is closer. Halfway or above rounds up. That is where the “5 or more, round up” rule comes from: the digit after the hundreds place is **7**.

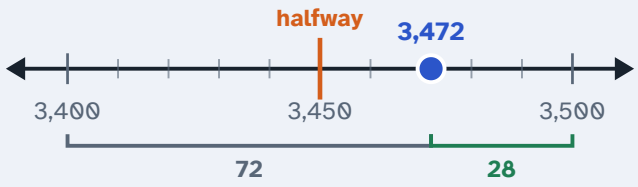


Step 4 of 5

3,472 ≥ 3,450 ⇒ round up

ROUND UP

Why? 3,472 is 72 above 3,400 and 28 below 3,500. It is closer to 3,500.



Step 5 of 5

3,472 ≈ 3,500

ANSWER

Why? So 3,472 rounded to the nearest hundred is **3,500**. The ≈ sign means “is about”.



Example 2

Round 485 to the nearest ten.

Step 1 of 5

Round 485 to the nearest ten

Why? The landmarks for “nearest ten” are the multiples of 10: 470, 480, 490, 500, … Rounding means picking whichever landmark is closest.



Each small tick is 1.

Step 2 of 5

485 is between 480 and 490

FIND THE TWO LANDMARKS

Why? Keep the digits up to the tens place and make the rest zeros: that gives the landmark just below, 480. One more ten gives the landmark just above, 490.



Step 3 of 5

Halfway between them: 485

HALFWAY POINT

Why? Halfway is the dividing line. Below halfway, the lower landmark is closer. Halfway or above rounds up. That is where the "5 or more, round up" rule comes from: the digit after the tens place is **5**.

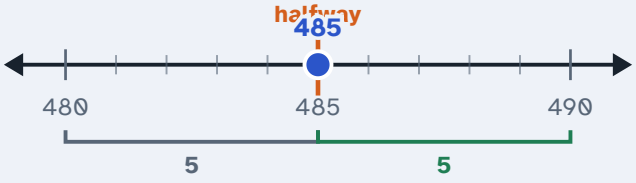


Step 4 of 5

485 ≥ 485 ⇒ round up

ROUND UP

Why? 485 is 5 above 480 and 5 below 490. It is exactly halfway, and the rule is that halfway rounds up.

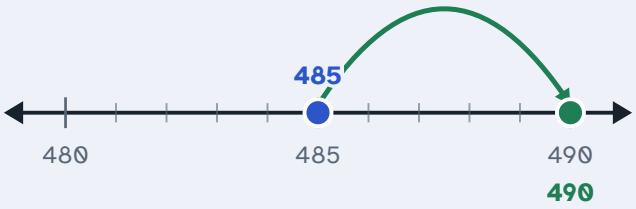


Step 5 of 5

485 ≈ 490

ANSWER

Why? So 485 rounded to the nearest ten is **490**. The ≈ sign means "is about".



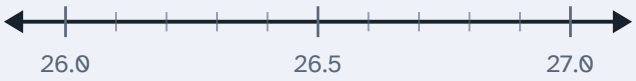
Example 3

Round 26.50 to the nearest whole number.

Step 1 of 5

Round 26.50 to the nearest whole number

Why? The landmarks for "nearest whole number" are the multiples of 1: 25, 26, 27, 28, ... Rounding means picking whichever landmark is closest.



Each small tick is 0.1.

Step 2 of 5

26.50 is between 26 and 27

FIND THE TWO LANDMARKS

> *Why?* Keep the digits up to the whole numbers place and make the rest zeros: that gives the landmark just below, 26. One more whole number gives the landmark just above, 27.



Step 3 of 5

Halfway between them: 27

HALFWAY POINT

> *Why?* Halfway is the dividing line. Below halfway, the lower landmark is closer. Halfway or above rounds up. That is where the "5 or more, round up" rule comes from: the digit after the whole numbers place is **5**.

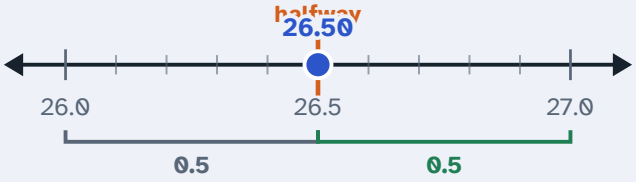


Step 4 of 5

26.50 ≥ 27 ⇒ round up

ROUND UP

> *Why?* 26.50 is 0.5 above 26 and 0.5 below 27. It is exactly halfway, and the rule is that halfway rounds up.



Step 5 of 5

26.50 ≈ 27

ANSWER

> *Why?* So 26.50 rounded to the nearest whole number is **27**. The ≈ sign means "is about".



2 Fractions and decimals on the line GRADES 4–6

A fraction is a point on the number line too. The bottom number cuts each whole into equal pieces; the top number counts the pieces.

Example 1

Where does $\frac{7}{4}$ go on the number line? Write it as a decimal.

Step 1 of 5

$\frac{7}{4}$

Why? The denominator, 4, says how many equal parts each whole is cut into. The numerator, 7, says how many of those parts to count.



Only the whole numbers are marked so far.

Step 2 of 5

Cut each whole into 4 equal parts

UNIT FRACTION

Why? Each small piece is $\frac{1}{4}$ of a whole. 4 of them make 1.



Each small tick is 1/4.

Step 3 of 5

Count 7 parts to the right of 0

POSITIVE: GO RIGHT

Why? Start at 0 and make 7 jumps of $\frac{1}{4}$ to the right.

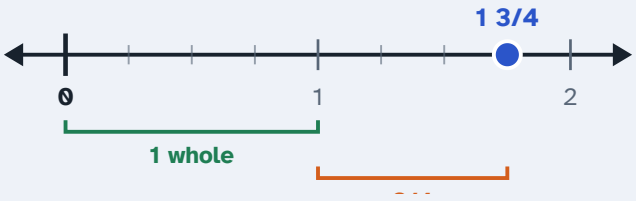


Step 4 of 5

$\frac{7}{4} = 1\frac{3}{4}$

MIXED NUMBER

Why? Every 4 parts make one whole. 7 ÷ 4 = 1 remainder 3, so it is 1 whole and $\frac{3}{4}$ more: just past 1.



Step 5 of 5

$\frac{7}{4} = 7 \div 4 = 1.75$

FRACTION BAR MEANS DIVIDE

Why? Divide the top by the bottom. Quarters: 1/4 = 0.25, so count by 0.25.



Example 2

Where does $\frac{3}{5}$ go on the number line? Write it as a decimal.

Step 1 of 5

$\frac{3}{5}$

Why? The denominator, 5, says how many equal parts each whole is cut into. The numerator, 3, says how many of those parts to count.



Only the whole numbers are marked so far.

Step 2 of 5

Cut each whole into 5 equal parts

UNIT FRACTION

Why? Each small piece is $\frac{1}{5}$ of a whole. 5 of them make 1.



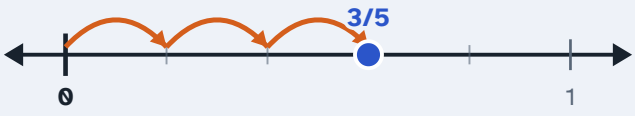
Each small tick is 1/5.

Step 3 of 5

Count 3 parts to the right of 0

POSITIVE: GO RIGHT

Why? Start at 0 and make 3 jumps of $\frac{1}{5}$ to the right.



Step 4 of 5

$\frac{3}{5}$ is less than one whole

PROPER FRACTION

Why? The top is smaller than the bottom, so it is only part of one whole: between 0 and 1.



Step 5 of 5

$\frac{3}{5} = 3 \div 5 = 0.6$

FRACTION BAR MEANS DIVIDE

Why? Divide the top by the bottom. Fifths: 1/5 = 0.2, so count by 0.2.



Example 3

Where does $-\frac{5}{3}$ go on the number line? Write it as a decimal.

Step 1 of 5

$-\frac{5}{3}$

Why? The denominator, 3, says how many equal parts each whole is cut into. The numerator, 5, says how many of those parts to count. The minus sign means count to the **left** of 0.



Only the whole numbers are marked so far.

Step 2 of 5

Cut each whole into 3 equal parts

UNIT FRACTION

Why? Each small piece is $\frac{1}{3}$ of a whole. 3 of them make 1.



Each small tick is 1/3.

Step 3 of 5

Count 5 parts to the left of 0

NEGATIVE: GO LEFT

Why? A negative fraction is the mirror image of the positive one: same distance from 0, other side. So count 5 jumps of $\frac{1}{3}$ to the left.

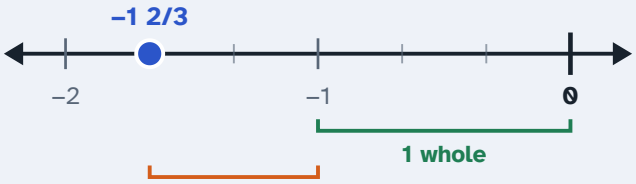


Step 4 of 5

$$\frac{5}{3} = 1\,\frac{2}{3}$$

MIXED NUMBER

Why? Every 3 parts make one whole. 5 ÷ 3 = 1 remainder 2, so it is 1 whole and $\frac{2}{3}$ more: just past −1, going left.



Step 5 of 5

$$-\frac{5}{3} = -5 \div 3 = -1.667\ldots$$

FRACTION BAR MEANS DIVIDE

Why? Divide the top by the bottom. This one never ends: the digits repeat forever, so round it to about −1.67.



3 Compare numbers, including negatives GRADE 6

Numbers get bigger as you go right. So whichever number sits further right is the greater one, even when both are negative.

Example 1

Fill in <, > or =: −7 ▢ −3

Step 1 of 4

−7 ? −3

> *Why?* On a number line, values increase from left to right. Put both numbers on the line; the one further right is greater.

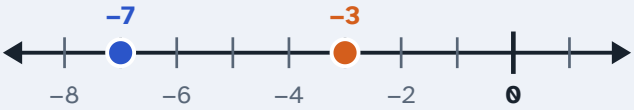


Step 2 of 4

Place −7 and −3

PLOT BOTH

> *Why?* −7 is 7 to the left of 0. −3 is 3 to the left of 0.

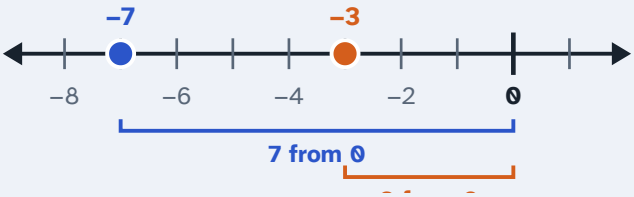


Step 3 of 4

−3 is further right ⇒ −7 < −3

RIGHT IS GREATER

> *Why?* Both are left of 0. The one **closer to 0** is greater. −7 is further from 0, so it is smaller, just like −7°F is colder than −3°F. Being "bigger-looking" without the minus makes a negative number *smaller*.

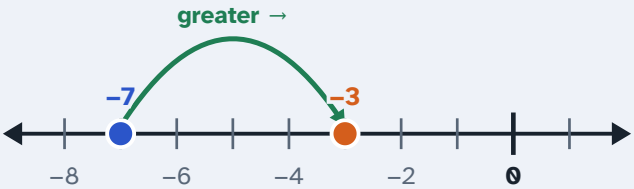


Step 4 of 4

−7 < −3

ANSWER

> *Why?* Read it left to right: "−7 is less than −3". The open side of the sign always faces the greater number.



Example 2

Fill in <, > or =: −2.5 ▢ −2.45

Step 1 of 5

−2.5 ? −2.45

> *Why?* On a number line, values increase from left to right. Put both numbers on the line; the one further right is greater.



Step 2 of 5

−2.50 and −2.45

SAME NUMBER OF DECIMAL PLACES

> *Why?* Write both with two decimal places so the digits line up: 2.50 and 2.45. Adding a zero on the end does not change a decimal.



Step 3 of 5

Place −2.5 and −2.45

PLOT BOTH

> *Why?* −2.5 is 2.5 to the left of 0. −2.45 is 2.45 to the left of 0.

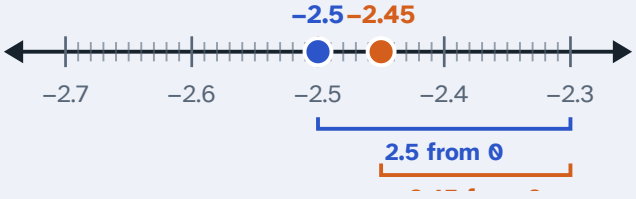


Step 4 of 5

−2.45 is further right ⇒ −2.5 < −2.45

RIGHT IS GREATER

> *Why?* Both are left of 0. The one **closer to 0** is greater. −2.5 is further from 0, so it is smaller, just like −7°F is colder than −3°F. Being "bigger-looking" without the minus makes a negative number *smaller*.

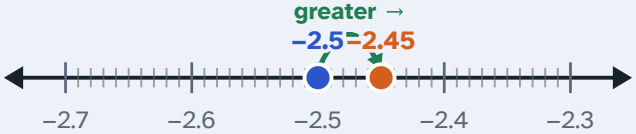


Step 5 of 5

−2.5 < −2.45

ANSWER

> *Why?* Read it left to right: "−2.5 is less than −2.45". The open side of the sign always faces the greater number.



Example 3

Fill in <, > or =: $-\frac{3}{4} \square -\frac{2}{3}$

Step 1 of 5

$-\frac{3}{4}$? $-\frac{2}{3}$

Why? On a number line, values increase from left to right. Put both numbers on the line; the one further right is greater.



Step 2 of 5

$-\frac{3}{4} \approx -0.75, \quad -\frac{2}{3} \approx -0.667$

CHANGE TO DECIMALS

Why? Fractions with different bottoms are hard to compare by eye. Dividing top by bottom turns each into a decimal you can place on the line.



Step 3 of 5

Place −3/4 and −2/3

PLOT BOTH

Why? −3/4 is 0.75 to the left of 0. −2/3 is 0.667 to the left of 0.

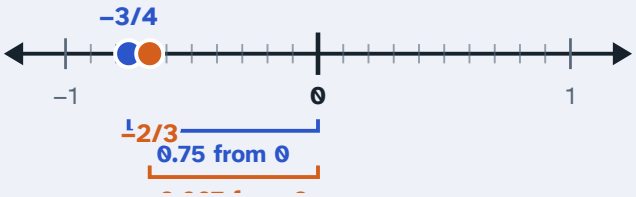


Step 4 of 5

−2/3 is further right ⇒ $-\frac{3}{4} < -\frac{2}{3}$

RIGHT IS GREATER

Why? Both are left of 0. The one **closer to 0** is greater. −3/4 is further from 0, so it is smaller, just like −7°F is colder than −3°F. Being "bigger-looking" without the minus makes a negative number *smaller*.

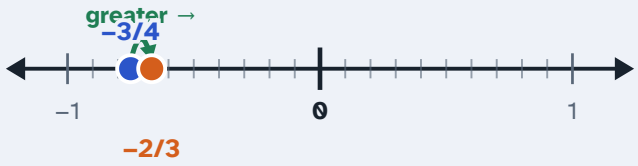


Step 5 of 5

$$-\frac{3}{4} < -\frac{2}{3}$$

ANSWER

Why? Read it left to right: “–3/4 is less than –2/3”. The open side of the sign always faces the greater number.



4 Opposites and absolute value GRADE 6

Opposites are mirror images across 0. The absolute value |a| is a number's distance from 0, so it is never negative.

Example 1

Find |−6|, and the opposite of −6.

Step 1 of 5

|−6| = ?

> *Why?* The bars | | mean absolute value: how far the number is from 0. A distance is never negative.



Step 2 of 5

Place −6

PLOT IT

> *Why?* −6 is left of 0.



Step 3 of 5

|−6| = 6

ABSOLUTE VALUE = DISTANCE FROM 0

> *Why?* Count the steps from 0 to −6: there are 6. The sign only tells you which side of 0; it doesn't change the distance.

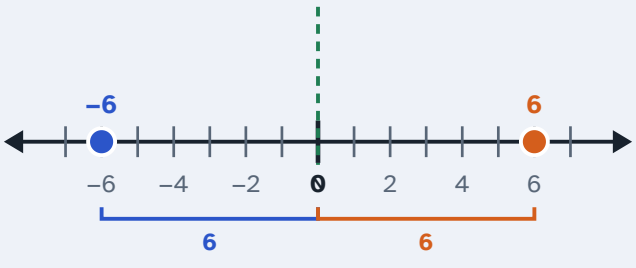


Step 4 of 5

Opposite of −6 = 6

OPPOSITES

> *Why?* The opposite is the mirror image across 0: same distance, other side. That is why |−6| = |6| = 6.

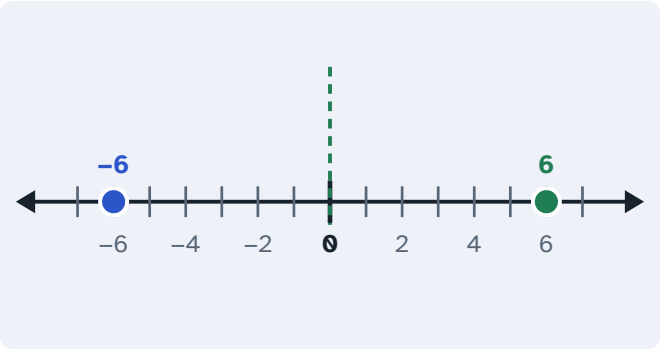


Step 5 of 5

|–6| = 6, opposite = 6

ANSWER

Why? Absolute value 6; opposite 6. Notice the opposite of a negative number is positive.



Example 2

Work out |–3| – |8|.

Step 1 of 5

|–3| – |8|

Why? Each pair of bars is its own little bracket. Turn each into a distance from 0 first, then subtract.



Step 2 of 5

|–3| = 3

ABSOLUTE VALUE = DISTANCE FROM 0

Why? –3 is 3 steps from 0.



Step 3 of 5

|8| = 8

ABSOLUTE VALUE = DISTANCE FROM 0

Why? 8 is 8 steps from 0.

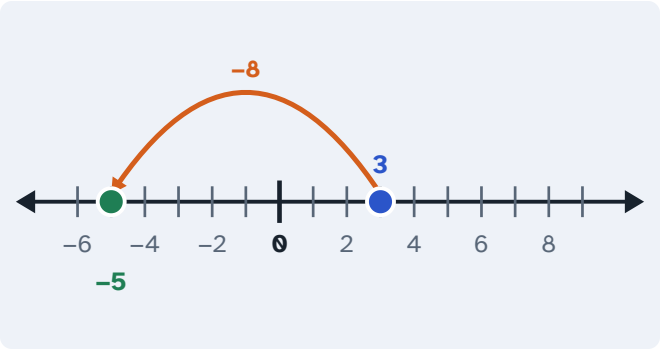


Step 4 of 5

3 − 8 = −5

SUBTRACT

Why? Start at 3 and move 8 to the left. It passes 0, so the answer is negative, even though both absolute values were positive.



Step 5 of 5

|−3| − |8| = −5

ANSWER

Why? Bars first, then the operation outside them.



Example 3

Work out |4 − 9|.

Step 1 of 4

|4 − 9|

Why? The bars work like brackets: do the inside first, then take the absolute value.

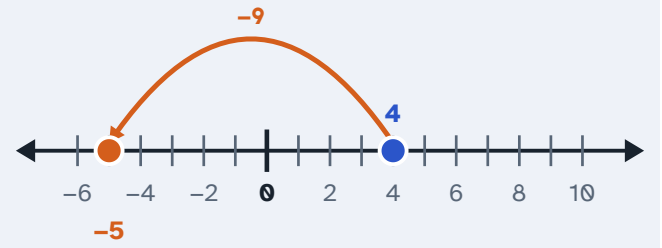


Step 2 of 4

4 − 9 = 4 + (−9) = −5

SUBTRACT: ADD THE OPPOSITE

Why? Subtracting 9 is the same as adding (−9): start at 4 and move 9 to the left.



Step 3 of 4

|–5| = 5

ABSOLUTE VALUE = DISTANCE FROM 0

Why? –5 is 5 steps from 0.

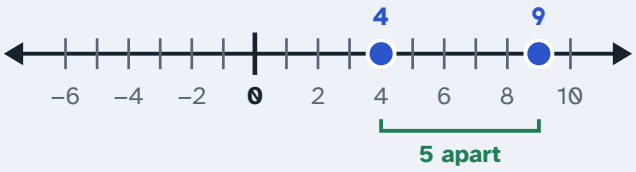


Step 4 of 4

|4 – 9| = 5

DISTANCE BETWEEN TWO NUMBERS

Why? Bonus: |4 – 9| is always the distance between 4 and 9 on the line. Count it: 5 steps.



Example 4

Work out –|–5|.

Step 1 of 4

–|–5|

Why? The minus sign is **outside** the bars. Bars first (like brackets), then take the opposite.



Step 2 of 4

|–5| = 5

ABSOLUTE VALUE = DISTANCE FROM 0

Why? –5 is 5 steps from 0.

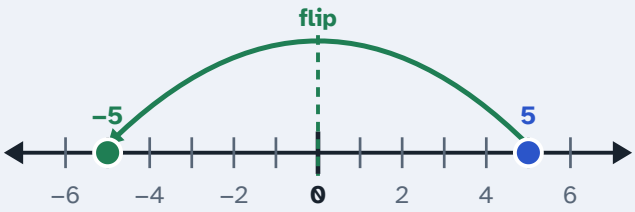


Step 3 of 4

−5

THEN THE OPPOSITE

Why? The outside minus flips 5 across 0 to −5. So −|anything| is never positive.



Step 4 of 4

−|−5| = −5

ANSWER

Why? Careful: |−5| is 5, but −|−5| is −5.



5 Adding negative numbers GRADE 7

Start at the first number. Adding a positive number moves you right; adding a negative number moves you left.

Example 1

−3 + 8 = ?

Step 1 of 4

−3 + 8 = ?

Why? Think of the number line as a path. Stand on the first number; each number you add is a move: positive means walk right, negative means walk left.



Step 2 of 4

Start at −3

START POINT

Why? −3 is 3 to the left of 0.

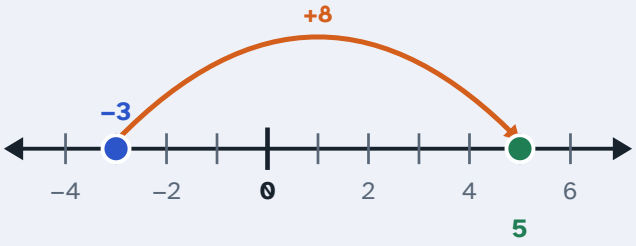


Step 3 of 4

−3 + 8 = 5

ADD A POSITIVE: MOVE RIGHT

Why? Move 8 to the right from −3. You cross 0 on the way.

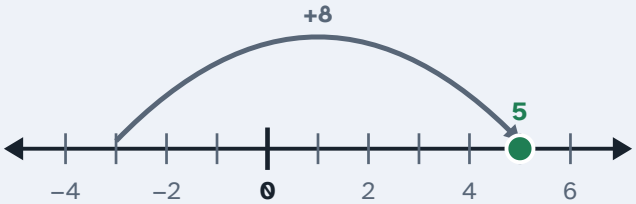


Step 4 of 4

−3 + 8 = 5

SHORTCUT RULE

Why? You landed on 5. The shortcut for next time: Different signs: subtract the sizes (8 − 3 = 5) and keep the sign of the bigger size (8).



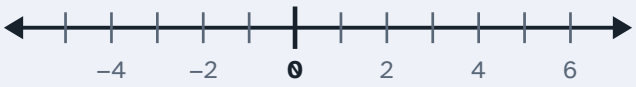
Example 2

5 + (−9) = ?

Step 1 of 4

5 + (−9) = ?

> *Why?* Think of the number line as a path. Stand on the first number; each number you add is a move: positive means walk right, negative means walk left.

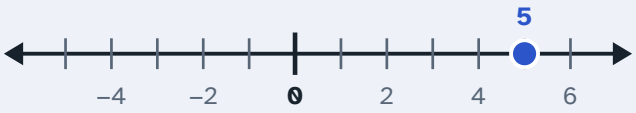


Step 2 of 4

Start at 5

START POINT

> *Why?* 5 is 5 to the right of 0.

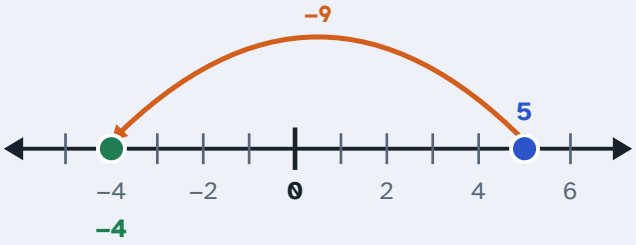


Step 3 of 4

5 + (−9) = −4

ADD A NEGATIVE: MOVE LEFT

> *Why?* Move 9 to the left from 5. You cross 0 on the way.

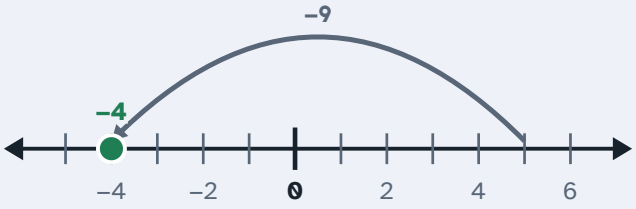


Step 4 of 4

5 + (−9) = −4

SHORTCUT RULE

> *Why?* You landed on −4. The shortcut for next time: Different signs: subtract the sizes (9 − 5 = 4) and keep the sign of the bigger size (−9).



Example 3

−4 + (−6) = ?

Step 1 of 4

−4 + (−6) = ?

Why? Think of the number line as a path. Stand on the first number; each number you add is a move: positive means walk right, negative means walk left.



Step 2 of 4

Start at −4

START POINT

Why? −4 is 4 to the left of 0.

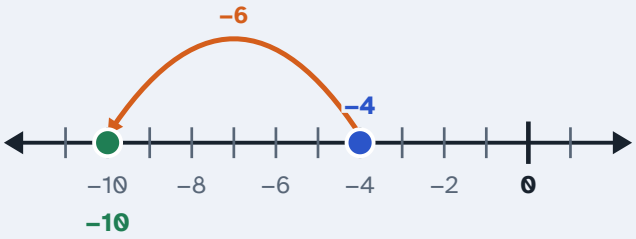


Step 3 of 4

−4 + (−6) = −10

ADD A NEGATIVE: MOVE LEFT

Why? Move 6 to the left from −4.

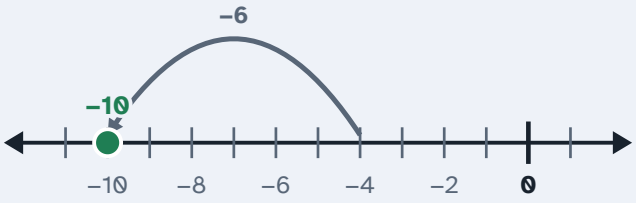


Step 4 of 4

−4 + (−6) = −10

SHORTCUT RULE

Why? You landed on −10. The shortcut for next time: Same signs: add the sizes (4 + 6 = 10) and keep the sign.



Example 4

7 + (−12) + 3 = ?

Step 1 of 5

7 + (−12) + 3 = ?

Why? Think of the number line as a path. Stand on the first number; each number you add is a move: positive means walk right, negative means walk left.



Step 2 of 5

Start at 7

START POINT

Why? 7 is 7 to the right of 0.

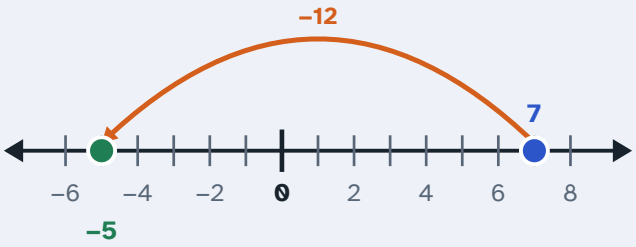


Step 3 of 5

7 + (−12) = −5

ADD A NEGATIVE: MOVE LEFT

Why? Move 12 to the left from 7. You cross 0 on the way.

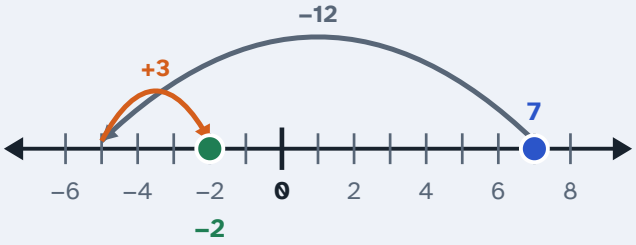


Step 4 of 5

−5 + 3 = −2

ADD A POSITIVE: MOVE RIGHT

Why? Move 3 to the right from −5.

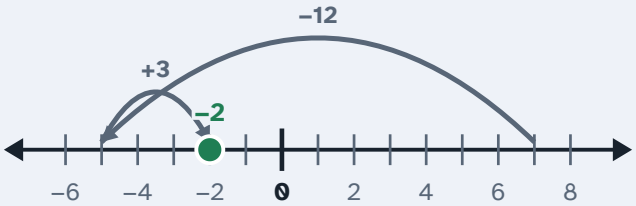


Step 5 of 5

7 + (−12) + 3 = −2

SHORTCUT RULE

Why? You landed on −2. The shortcut for next time: Group them: positives make 10, negatives make −12. Then 10 + (−12) = −2.



6 Subtracting negative numbers GRADE 7

Subtracting means adding the opposite. Taking away a negative moves you right, the way paying off a debt leaves you richer.

Example 1

3 − (−5) = ?

Step 1 of 6

3 − (−5)

Why? Subtraction has a trick that always works: change it into addition. Subtracting a number is the same as adding its opposite.



Step 2 of 6

⇒ **3 + 5**

ADD THE OPPOSITE (KEEP · CHANGE · CHANGE)

Why? Keep the first number. Change each − to +. Change the sign of the number after it. So − (−5) becomes + 5: two minuses make a plus.



Step 3 of 6

Start at 3

START POINT

Why? Stand on 3.

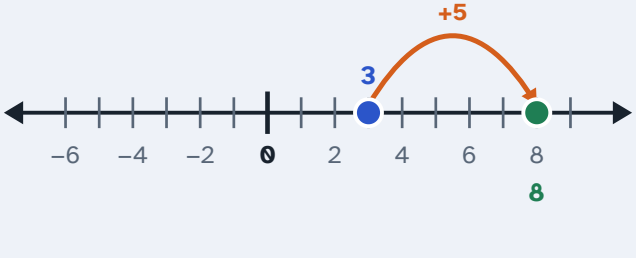


Step 4 of 6

3 + 5 = 8

SUBTRACT A NEGATIVE: MOVE RIGHT

Why? Taking away −5 is like removing a debt of 5: you end up 5 higher. Move right.

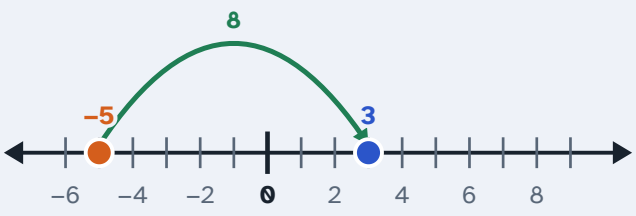


Step 5 of 6

Check: from −5 to 3 is 8 right ⇒ 8

DIFFERENCE = HOW FAR FROM B TO A

Why? Another way to see 3 − (−5): it asks "how far is it from −5 to 3?" Going right counts as positive, going left as negative. It agrees: 8.



Step 6 of 6

3 − (−5) = 8

ANSWER

Why? Rewrite as adding the opposite, then walk the number line.



Example 2

−4 − 6 = ?

Step 1 of 6

−4 − 6

Why? Subtraction has a trick that always works: change it into addition. Subtracting a number is the same as adding its opposite.



Step 2 of 6

⇒ −4 + (−6)

ADD THE OPPOSITE (KEEP · CHANGE · CHANGE)

Why? Keep the first number. Change each − to +. Change the sign of the number after it.

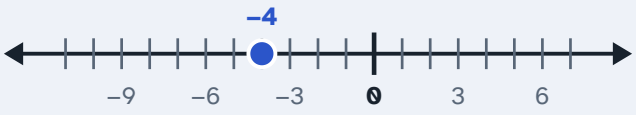


Step 3 of 6

Start at −4

START POINT

Why? Stand on −4.

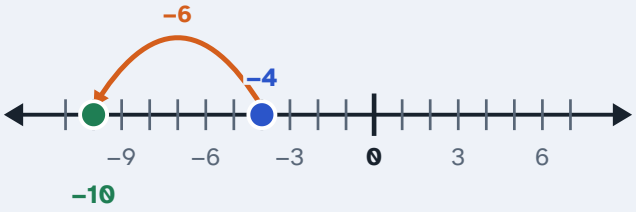


Step 4 of 6

−4 + (−6) = −10

SUBTRACT A POSITIVE: MOVE LEFT

Why? Taking away 6 moves you 6 to the left.

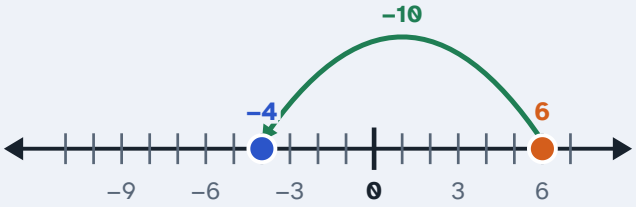


Step 5 of 6

Check: from 6 to −4 is 10 left ⇒ −10

DIFFERENCE = HOW FAR FROM B TO A

Why? Another way to see −4 − 6: it asks “how far is it from 6 to −4?” Going right counts as positive, going left as negative. It agrees: −10.



Step 6 of 6

−4 − 6 = −10

ANSWER

Why? Rewrite as adding the opposite, then walk the number line.



Example 3

−2 − (−9) = ?

Step 1 of 6

−2 − (−9)

> *Why?* Subtraction has a trick that always works: change it into addition. Subtracting a number is the same as adding its opposite.



Step 2 of 6

⇒ −2 + 9

`ADD THE OPPOSITE (KEEP · CHANGE · CHANGE)`

> *Why?* Keep the first number. Change each − to +. Change the sign of the number after it. So − (−9) becomes + 9: two minuses make a plus.



Step 3 of 6

Start at −2

`START POINT`

> *Why?* Stand on −2.

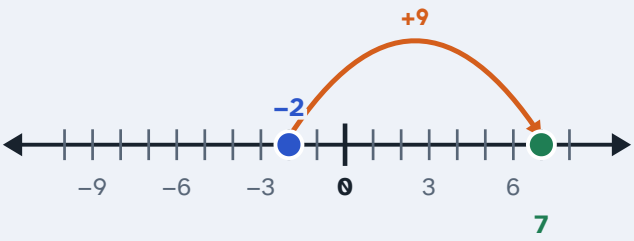


Step 4 of 6

−2 + 9 = 7

`SUBTRACT A NEGATIVE: MOVE RIGHT`

> *Why?* Taking away −9 is like removing a debt of 9: you end up 9 higher. Move right.

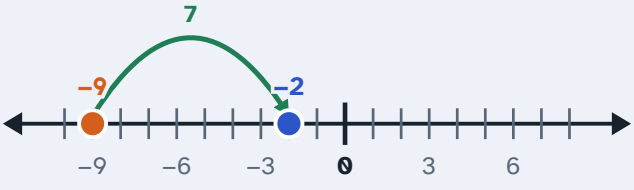


Step 5 of 6

Check: from −9 to −2 is 7 right ⇒ 7

`DIFFERENCE = HOW FAR FROM B TO A`

> *Why?* Another way to see −2 − (−9): it asks “how far is it from −9 to −2?” Going right counts as positive, going left as negative. It agrees: 7.



Step 6 of 6

–2 – (–9) = 7

ANSWER

Why? Rewrite as adding the opposite, then walk the number line.



Example 4

5 – (–3) – 10 = ?

Step 1 of 6

5 – (–3) – 10

Why? Subtraction has a trick that always works: change it into addition. Subtracting a number is the same as adding its opposite.



Step 2 of 6

⇒ **5 + 3 + (–10)**

ADD THE OPPOSITE (KEEP · CHANGE · CHANGE)

Why? Keep the first number. Change each – to +. Change the sign of the number after it. So – (–3) becomes + 3: two minuses make a plus.



Step 3 of 6

Start at 5

START POINT

Why? Stand on 5.

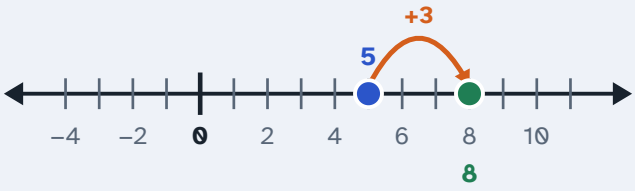


Step 4 of 6

5 + 3 = 8

SUBTRACT A NEGATIVE: MOVE RIGHT

Why? Taking away −3 is like removing a debt of 3: you end up 3 higher. Move right.

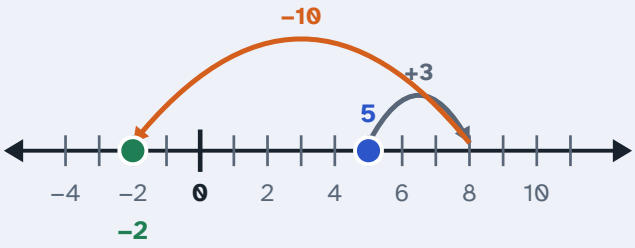


Step 5 of 6

8 + (−10) = −2

SUBTRACT A POSITIVE: MOVE LEFT

Why? Taking away 10 moves you 10 to the left.



Step 6 of 6

5 − (−3) − 10 = −2

ANSWER

Why? Rewrite as adding the opposite, then walk the number line.



7 Multiplying and dividing negatives GRADES 7–8

Multiplying is repeated jumps. A negative sign means “flip to the other side of 0”. Two flips bring you back, which is why negative × negative is positive.

Example 1

3 × (−4) = ?

Step 1 of 3

3 × (−4)

Why? 3 × (−4) means 3 jumps of size (−4), starting at 0.



Step 2 of 3

3 jumps of (−4) ⇒ −12

REPEATED ADDITION

Why? Each jump is 4 to the left. 3 of them land on −12.



Step 3 of 3

3 × (−4) = −12

SIGN RULE

Why? Same signs give a positive answer; different signs give a negative answer. Here the signs are different, so negative: 3 × 4 = 12, so the answer is −12.



Example 2

(−2) × (−5) = ?

Step 1 of 4

(−2) × (−5)

Why? (−2) × (−5) means 2 jumps of size (−5), starting at 0. The first number is negative, so after the jumps, flip to the other side of 0.



Step 2 of 4

2 jumps of (−5) ⇒ −10

REPEATED ADDITION

Why? Each jump is 5 to the left. 2 of them land on −10.

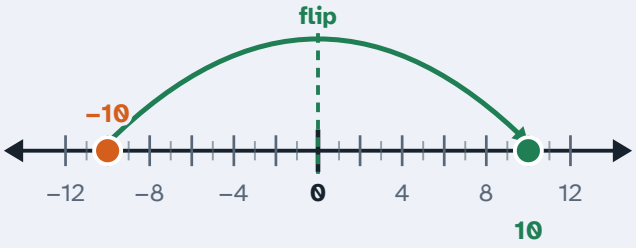


Step 3 of 4

Opposite of −10 = 10

NEGATIVE = FLIP ACROSS 0

Why? Multiplying by a negative number means “the opposite of”. Flip −10 across 0 to 10. Two negatives = two flips = back to the positive side.



Step 4 of 4

(−2) × (−5) = 10

SIGN RULE

Why? Same signs give a positive answer; different signs give a negative answer. Here the signs are the same, so positive: 2 × 5 = 10, so the answer is 10.



Example 3

(−12) ÷ 3 = ?

Step 1 of 5

(−12) ÷ 3

Why? Division undoes multiplication. (−12) ÷ 3 asks: “3 times what makes −12?”



Step 2 of 5

3 × ? = −12

RELATED MULTIPLICATION FACT

Why? Find the missing number. First its size, then its sign.

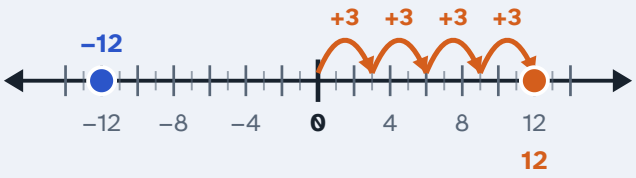


Step 3 of 5

12 ÷ 3 = 4

SIZE FIRST

Why? 4 jumps of 3 from 0 land on 12.

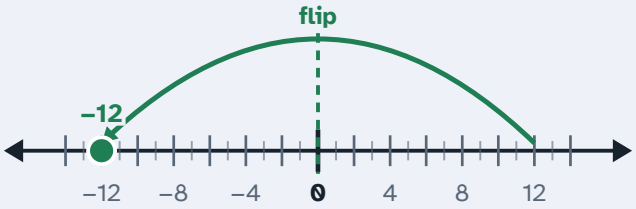


Step 4 of 5

Landed on the opposite ⇒ flip ⇒ ? = −4

SIGN RULE

Why? The jumps reach 12, the opposite of −12. A flip is needed, so the answer is negative. (Different signs → negative.)



Step 5 of 5

Check: 3 × (−4) = −12 ✓

ANSWER

Why? (−12) ÷ 3 = −4. Division follows the same sign rule as multiplication.



Example 4

$(-2)^3$ = ?

Step 1 of 5

$(-2)^3$

Why? The exponent 3 means multiply 3 times by (−2). Start at 1. Each multiply by a negative flips you to the other side of 0 and stretches by 2.

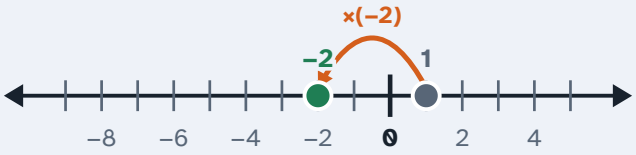


Step 2 of 5

$(-2)^1 = -2$

ODD NUMBER OF FLIPS: NEGATIVE

Why? 1 × (−2) = −2. That is an odd number of flips, so it lands left of 0.

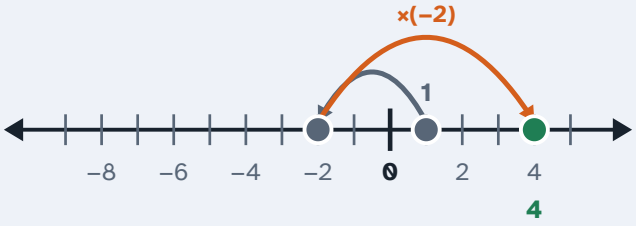


Step 3 of 5

$(-2)^2 = 4$

EVEN NUMBER OF FLIPS: POSITIVE

Why? −2 × (−2) = 4. An even number of flips cancels out, so it lands right of 0.

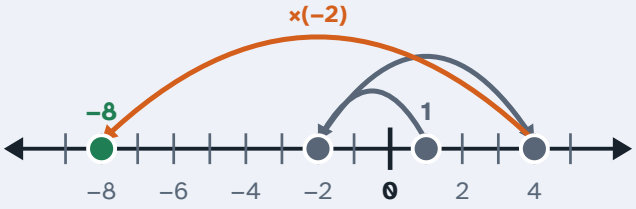


Step 4 of 5

$(-2)^3 = -8$

ODD NUMBER OF FLIPS: NEGATIVE

Why? 4 × (−2) = −8. That is an odd number of flips, so it lands left of 0.



Step 5 of 5

$(-2)^3 = -8$

PATTERN

Why? A negative number to an **even** power is positive; to an **odd** power it is negative. Here 3 is odd.



Example 5

$-3^2 = ?$

Step 1 of 4

-3^2

> *Why?* No brackets! The little ^{2} belongs only to the 3. Order of operations: exponents come before the minus sign.



Step 2 of 4

$3^2 = 3 \times 3 = 9$

EXPONENT FIRST

> *Why?* Square the 3 on its own.

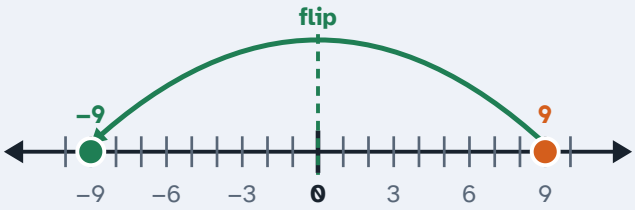


Step 3 of 4

-9

THEN THE OPPOSITE

> *Why?* Now apply the minus: the opposite of 9 is −9.



Step 4 of 4

$-3^2 = -9$, but $(-3)^2 = 9$

BRACKETS MATTER

> *Why?* With brackets, the −3 is squared as a whole: (−3) × (−3) = 9. Without brackets only the 3 is squared. Calculators follow this rule too.



8 Distance and midpoint GRADES 6–8

Distance is how many units apart two points are, and it is never negative. The midpoint is exactly halfway, the average of the two ends.

Example 1

How far apart are −5 and 7?

Step 1 of 5

How far apart are −5 and 7?

Why? Distance is the number of units between the two points. It is never negative, whichever order you name them.



Step 2 of 5

Place −5 and 7

PLOT BOTH

Why? They are on opposite sides of 0.

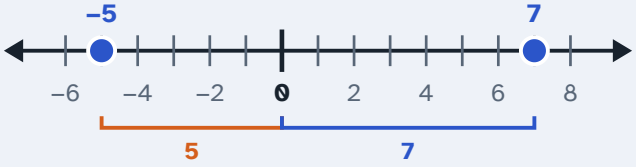


Step 3 of 5

5 + 7 = 12

OPPOSITE SIDES OF 0: ADD

Why? Walk from −5 to 0 (5 steps), then from 0 to 7 (7 steps). Add the two pieces.

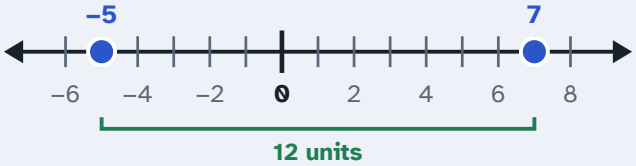


Step 4 of 5

|−5 − 7| = |−12| = 12

DISTANCE = |A − B|

Why? One rule that always works: subtract in either order, then take the absolute value. |7 − (−5)| = |12| = 12 too.

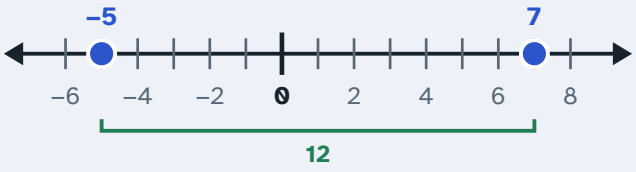


Step 5 of 5

Distance = 12

ANSWER

Why? –5 and 7 are 12 units apart.



Example 2

How far apart are –11 and –4?

Step 1 of 5

How far apart are –11 and –4?

Why? Distance is the number of units between the two points. It is never negative, whichever order you name them.



Step 2 of 5

Place –11 and –4

PLOT BOTH

Why? They are on the same side of 0.



Step 3 of 5

11 – 4 = 7

SAME SIDE OF 0: SUBTRACT

Why? Both are on the negative side. The farther one is 11 from 0, the nearer one 4. The gap between them is the difference.

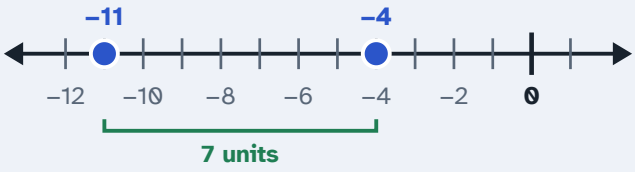


Step 4 of 5

|–11 – (–4)| = |–7| = 7

DISTANCE = |A – B|

Why? One rule that always works: subtract in either order, then take the absolute value. |–4 – (–11)| = |7| = 7 too.



Step 5 of 5

Distance = 7

ANSWER

Why? –11 and –4 are 7 units apart.



Example 3

What number is exactly halfway between –9 and 4?

Step 1 of 5

What number is exactly halfway between –9 and 4?

Why? The midpoint is the same distance from both ends. Find the whole distance, cut it in half, and walk that far from one end.



Step 2 of 5

Place –9 and 4

PLOT BOTH

Why? The left end is –9, the right end is 4.

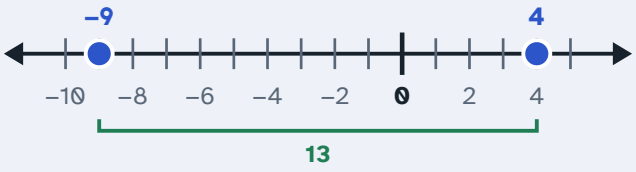


Step 3 of 5

Distance = |4 − (−9)| = 13

DISTANCE = |A − B|

Why? From −9 to 4 is 13 units.

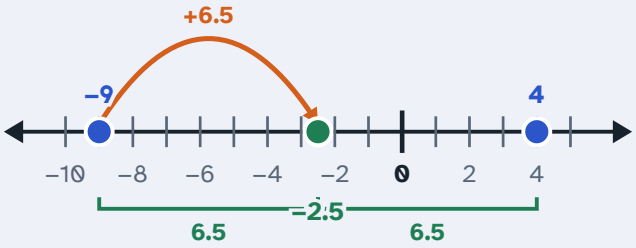


Step 4 of 5

Half of 13 = 6.5 ⇒ −9 + 6.5 = −2.5

WALK HALF THE DISTANCE

Why? Start at the left end, −9, and move 6.5 to the right. The distance is odd, so the midpoint lands between two whole numbers.



Step 5 of 5

Check: $\frac{-9 + 4}{2} = \frac{-5}{2} = -2.5$ ✓

MIDPOINT = AVERAGE

Why? Adding the ends and dividing by 2 is the same thing, and it is the formula used later in coordinate geometry: midpoint = $\frac{a + b}{2}$.



Square roots on the number line GRADE 8

Most square roots are not whole numbers, but you can still pin them down: trap them between two perfect squares, then zoom in.

Example 1

Place $\sqrt{50}$ on the number line and estimate it to one decimal place.

Step 1 of 5

$\sqrt{50} \approx$?

Why? $\sqrt{50}$ is the number that, multiplied by itself, makes 50. 50 is not a perfect square, so $\sqrt{50}$ is not a whole number.



Step 2 of 5

$7^2 = 49 < 50 < 64 = 8^2$

TRAP BETWEEN PERFECT SQUARES

Why? 50 sits between the perfect squares 49 and 64, so $\sqrt{50}$ sits between 7 and 8.



$\sqrt{50}$ is somewhere on the blue piece.

Step 3 of 5

$50 - 49 = 1, \quad 64 - 50 = 14$

WHICH END IS CLOSER?

Why? 50 is 1 above 49 and 14 below 64, so $\sqrt{50}$ is closer to 7.



Zoomed in: each tick is one tenth.

Step 4 of 5

$\sqrt{50} \approx 7 + \frac{1}{15} \approx 7.07$

ESTIMATE BY FRACTION OF THE GAP

Why? From 49 to 64 is a gap of 15. 50 is 1 of the way along, so $\sqrt{50}$ is about $\frac{1}{15}$ of the way from 7 to 8. (This guess runs a little low, because squares spread out as they grow.)



Step 5 of 5

√50 ≈ 7.1 **(calculator: 7.071…)**

ANSWER

Why? To one decimal place, √50 ≈ 7.1. The digits never end or repeat: √50 is an **irrational** number, but it still has an exact spot on the line.



Example 2

Place √20 on the number line and estimate it to one decimal place.

Step 1 of 5

√20 ≈ ?

Why? √20 is the number that, multiplied by itself, makes 20. 20 is not a perfect square, so √20 is not a whole number.



Step 2 of 5

$4^2 = 16 < 20 < 25 = 5^2$

TRAP BETWEEN PERFECT SQUARES

Why? 20 sits between the perfect squares 16 and 25, so √20 sits between 4 and 5.



√20 is somewhere on the blue piece.

Step 3 of 5

20 − 16 = 4, 25 − 20 = 5

WHICH END IS CLOSER?

Why? 20 is 4 above 16 and 5 below 25, so √20 is closer to 4.



Zoomed in: each tick is one tenth.

Step 4 of 5

$\sqrt{20} \approx 4 + \frac{4}{9} \approx 4.44$

ESTIMATE BY FRACTION OF THE GAP

Why? From 16 to 25 is a gap of 9. 20 is 4 of the way along, so $\sqrt{20}$ is about $\frac{4}{9}$ of the way from 4 to 5. (This guess runs a little low, because squares spread out as they grow.)

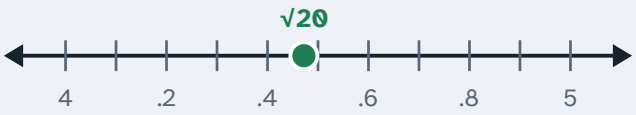


Step 5 of 5

$\sqrt{20} \approx 4.5$ **(calculator: 4.472…)**

ANSWER

Why? To one decimal place, $\sqrt{20} \approx 4.5$. The digits never end or repeat: $\sqrt{20}$ is an **irrational** number, but it still has an exact spot on the line.



Example 3

Place $\sqrt{140}$ on the number line and estimate it to one decimal place.

Step 1 of 6

$\sqrt{140} \approx$?

Why? $\sqrt{140}$ is the number that, multiplied by itself, makes 140. 140 is not a perfect square, so $\sqrt{140}$ is not a whole number.



Step 2 of 6

$11^2 = 121 < 140 < 144 = 12^2$

TRAP BETWEEN PERFECT SQUARES

Why? 140 sits between the perfect squares 121 and 144, so $\sqrt{140}$ sits between 11 and 12.



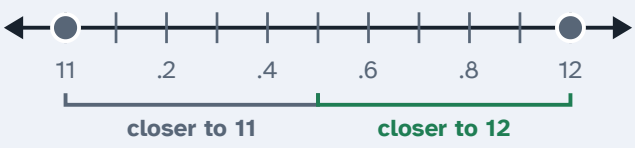
$\sqrt{140}$ is somewhere on the blue piece.

Step 3 of 6

$140 - 121 = 19, \quad 144 - 140 = 4$

WHICH END IS CLOSER?

Why? 140 is 19 above 121 and 4 below 144, so √140 is closer to 12.



Zoomed in: each tick is one tenth.

Step 4 of 6

$\sqrt{140} \approx 11 + \frac{19}{23} \approx 11.83$

ESTIMATE BY FRACTION OF THE GAP

Why? From 121 to 144 is a gap of 23. 140 is 19 of the way along, so √140 is about $\frac{19}{23}$ of the way from 11 to 12. (This guess runs a little low, because squares spread out as they grow.)



Step 5 of 6

$11.8^2 = 139.24, \quad 11.9^2 = 141.61$

CHECK BY SQUARING

Why? Square the tenths on each side: 139.24 < 140 < 141.61. So √140 is between 11.8 and 11.9, and closer to 11.8 because 140 is nearer to 139.24.



Step 6 of 6

$\sqrt{140} \approx 11.8$ (calculator: 11.832…)

ANSWER

Why? To one decimal place, √140 ≈ 11.8. The digits never end or repeat: √140 is an **irrational** number, but it still has an exact spot on the line.



Example 4

Place $-\sqrt{30}$ on the number line and estimate it to one decimal place.

Step 1 of 6

$-\sqrt{30} \approx ?$

Why? $\sqrt{30}$ is the number that, multiplied by itself, makes 30. 30 is not a perfect square, so $\sqrt{30}$ is not a whole number. The minus outside means: find $\sqrt{30}$, then take its opposite.



Step 2 of 6

$5^2 = 25 < 30 < 36 = 6^2$

TRAP BETWEEN PERFECT SQUARES

Why? 30 sits between the perfect squares 25 and 36, so $\sqrt{30}$ sits between 5 and 6.



$\sqrt{30}$ is somewhere on the blue piece.

Step 3 of 6

$30 - 25 = 5, \quad 36 - 30 = 6$

WHICH END IS CLOSER?

Why? 30 is 5 above 25 and 6 below 36, so $\sqrt{30}$ is closer to 5.



Zoomed in: each tick is one tenth.

Step 4 of 6

$\sqrt{30} \approx 5 + \frac{5}{11} \approx 5.45$

ESTIMATE BY FRACTION OF THE GAP

Why? From 25 to 36 is a gap of 11. 30 is 5 of the way along, so $\sqrt{30}$ is about $\frac{5}{11}$ of the way from 5 to 6. (This guess runs a little low, because squares spread out as they grow.)

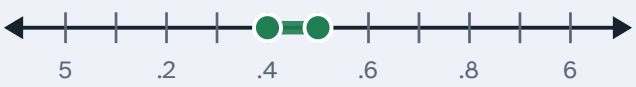


Step 5 of 6

$5.4^2 = 29.16$, $5.5^2 = 30.25$

CHECK BY SQUARING

Why? Square the tenths on each side: 29.16 < 30 < 30.25. So √30 is between 5.4 and 5.5, and closer to 5.5 because 30 is nearer to 30.25.

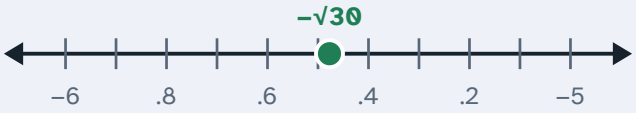


Step 6 of 6

$-\sqrt{30} \approx -5.5$ (calculator: −5.477…)

THEN THE OPOSITE

Why? √30 ≈ 5.5, so −√30 is its mirror image: about −5.5, between −6 and −5.



10 Solve and graph inequalities GRADES 7–9

An inequality has a whole stretch of answers. Solve it like an equation, with one twist: multiplying or dividing by a negative number flips the sign.

Example 1

Solve and graph: $x + 4 < 1$

Step 1 of 5

$x + 4 < 1$

> *Why?* Get x alone, just like solving an equation. Adding, subtracting, or multiplying/dividing by a **positive** keeps the sign the same. Multiplying or dividing by a **negative** flips it.



Step 2 of 5

$\Rightarrow x < 1 \ (-4) = -3$

SUBTRACT FROM BOTH SIDES (SIGN STAYS)

> *Why?* Take 4 away from both sides. That slides both sides along the line by the same amount, so their order doesn't change.



Step 3 of 5

$x < -3$

OPEN CIRCLE: NOT INCLUDED

> *Why?* < doesn't include the boundary, so draw an **open** circle at −3. Then shade toward the numbers that work: less, so shade left.

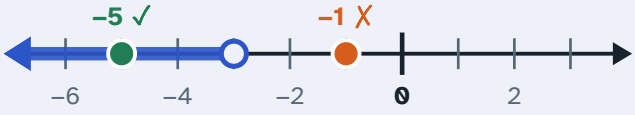


Step 4 of 5

Test −5: $-5 + 4 = -1 < 1$ ✓ Test −1: $3 < 1$ ✗

TEST A POINT

> *Why?* Pick a number in the shaded part and put it into the **original** inequality: it works. A number outside doesn't. That confirms the direction.



Step 5 of 5

$x < -3$ or $(-\infty, -3)$

INTERVAL NOTATION

Why? Interval notation lists the left end, then the right end. A square bracket [ ] means the end is included; a round bracket ( ) means it isn't. Infinity always gets a round bracket, because you can never reach it.



Example 2

Solve and graph: $3x - 2 \geq 10$

Step 1 of 6

$3x - 2 \geq 10$

Why? Get x alone, just like solving an equation. Adding, subtracting, or multiplying/dividing by a **positive** keeps the sign the same. Multiplying or dividing by a **negative** flips it.



Step 2 of 6

$\Rightarrow 3x \geq 10 \ \textcircled{+2} = 12$

ADD TO BOTH SIDES (SIGN STAYS)

Why? Add 2 to both sides. That slides both sides along the line by the same amount, so their order doesn't change.



Step 3 of 6

$\Rightarrow x \geq \frac{12}{3} = 4$

DIVIDE BY A POSITIVE (SIGN STAYS)

Why? Divide both sides by 3. A positive number keeps the order, so the sign stays ≥.



Step 4 of 6

$x \geq 4$

CLOSED CIRCLE: INCLUDED

Why? ≥ includes the boundary, so **fill in** the circle at 4. Then shade toward the numbers that work: greater, so shade right.

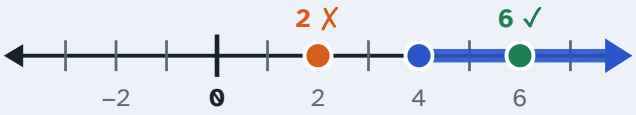


Step 5 of 6

Test 6: 3·6 − 2 = 16 ≥ 10 ✓ Test 2: 4 ≥ 10 ✗

TEST A POINT

Why? Pick a number in the shaded part and put it into the **original** inequality: it works. A number outside doesn't. That confirms the direction.



Step 6 of 6

$x \geq 4$ or $[4, \infty)$

INTERVAL NOTATION

Why? Interval notation lists the left end, then the right end. A square bracket [ ] means the end is included; a round bracket ( ) means it isn't. Infinity always gets a round bracket, because you can never reach it.



Example 3

Solve and graph: $-2x + 5 > 13$

Step 1 of 6

$-2x + 5 > 13$

Why? Get x alone, just like solving an equation. Adding, subtracting, or multiplying/dividing by a **positive** keeps the sign the same. Multiplying or dividing by a **negative** flips it.



Step 2 of 6

$\Rightarrow -2x > 13 \ (-5) = 8$

SUBTRACT FROM BOTH SIDES (SIGN STAYS)

Why? Take 5 away from both sides. That slides both sides along the line by the same amount, so their order doesn't change.

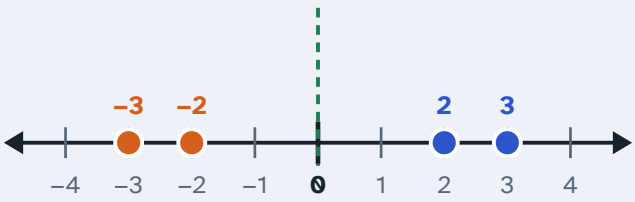


Step 3 of 6

$\Rightarrow x \ (<) \ \frac{8}{-2} = -4$

DIVIDE BY A NEGATIVE: FLIP THE SIGN

Why? Divide both sides by −2. Dividing by a negative mirrors every number across 0, which reverses their order. Look: 2 < 3, but −2 > −3. So > becomes <.



2 is left of 3, but −2 is right of −3: the mirror reverses the order.

Step 4 of 6

$x < -4$

OPEN CIRCLE: NOT INCLUDED

Why? < doesn't include the boundary, so draw an **open** circle at −4. Then shade toward the numbers that work: less, so shade left.



Step 5 of 6

Test −6: $-2\cdot(-6) + 5 = 17 > 13$ ✓ Test −2: $9 > 13$ ✗

TEST A POINT

Why? Pick a number in the shaded part and put it into the **original** inequality: it works. A number outside doesn't. That confirms the direction (and that the flip was right).



Step 6 of 6

$x < -4$ or $(-\infty, -4)$

INTERVAL NOTATION

Why? Interval notation lists the left end, then the right end. A square bracket [ ] means the end is included; a round bracket ( ) means it isn't. Infinity always gets a round bracket, because you can never reach it.



11 Absolute value equations and inequalities

GRADES 9–11

|x – c| is the distance between x and c. Reading it as a distance turns every absolute value problem into a picture.

Example 1

Solve: $|x - 4| < 3$

Step 1 of 5

$|x - 4| < 3$

Why? Read $|x - 4|$ as a distance: "how far x is from 4".



Step 2 of 5

The distance from x to 4 is less than 3

ABSOLUTE VALUE = DISTANCE

Why? Every answer is a point whose distance from 4 is less than 3.

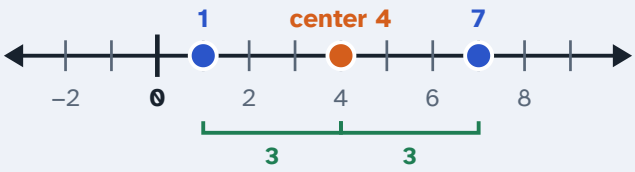


Step 3 of 5

$4 - 3 = 1, \quad 4 + 3 = 7$

GO R EACH WAY FROM THE CENTER

Why? The points exactly 3 away are 3 to the left and 3 to the right of 4. These are the boundary points.

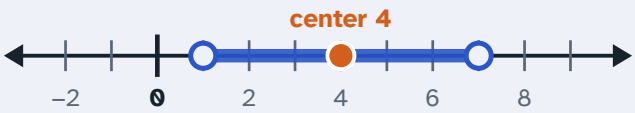


Step 4 of 5

$1 < x < 7$

LESS THAN: BETWEEN (AND)

Why? Closer than 3 means staying between the boundaries: x is more than 1 **and** less than 7. The ends don't count, so the circles are open.



Step 5 of 5

(1, 7) Check $x = 4$: $|4 - 4| = 0 < 3$ ✓

INTERVAL NOTATION + CHECK

Why? One piece, written [left, right] or (left, right). Testing a point from the answer in the original problem confirms it.



Example 2

Solve: $|x + 2| \geq 5$

Step 1 of 5

$|x + 2| \geq 5$

Why? Read $|x + 2|$ as a distance: "how far x is from -2". Note $x + 2 = x - (-2)$, so the center is -2.



Step 2 of 5

The distance from x to -2 is at least 5

ABSOLUTE VALUE = DISTANCE

Why? $x + 2$ is the same as $x - (-2)$, so the center is -2. Every answer is a point whose distance from -2 is at least 5.

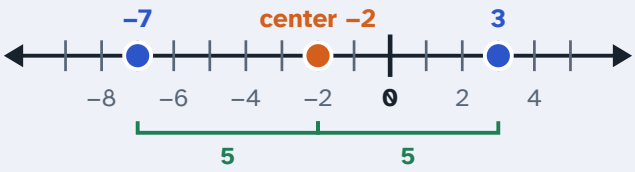


Step 3 of 5

$-2 - 5 = -7$, $-2 + 5 = 3$

GO R EACH WAY FROM THE CENTER

Why? The points exactly 5 away are 5 to the left and 5 to the right of -2. These are the boundary points.



Step 4 of 5

$x \le -7$ or $x \ge 3$

GREATER THAN: OUTSIDE (OR)

Why? Farther than 5 means going past a boundary on either side: left of –7 **or** right of 3. The ends count (filled circles).



Step 5 of 5

$(-\infty, -7] \cup [3, \infty)$ Check $x = 4$: $|4 - (-2)| = 6 \ge 5$ ✓

INTERVAL NOTATION + CHECK

Why? Two pieces joined by ∪, which means "or". Testing a point from the answer in the original problem confirms it.



Example 3

Solve: $|x - 1| = 4$

Step 1 of 5

$|x - 1| = 4$

Why? Read $|x - 1|$ as a distance: "how far x is from 1".



Step 2 of 5

The distance from x to 1 is exactly 4

ABSOLUTE VALUE = DISTANCE

Why? Every answer is a point whose distance from 1 is exactly 4.

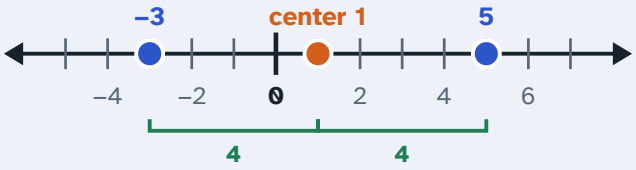


Step 3 of 5

1 − 4 = −3, 1 + 4 = 5

GO R EACH WAY FROM THE CENTER

Why? The points exactly 4 away are 4 to the left and 4 to the right of 1. These are the boundary points.



Step 4 of 5

$x = -3$ or $x = 5$

EQUALS: JUST THE TWO POINTS

Why? Only two numbers are exactly 4 away.



Step 5 of 5

{−3, 5} Check $x = -3$: $|-3 - 1| = 4 = 4$ ✓

INTERVAL NOTATION + CHECK

Why? Curly brackets list separate answers. Testing a point from the answer in the original problem confirms it.



Example 4

Solve: $|x - 3| + 5 \leq 7$

Step 1 of 6

Solve: $|x - 3| + 5 \leq 7$

Why? First get the absolute value on its own, like any other term.

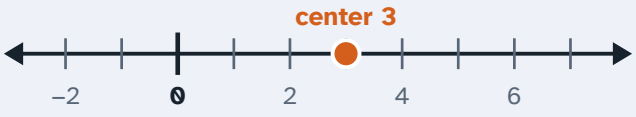


Step 2 of 6

⇒ $|x - 3| \le 2$

SUBTRACT 5 ON BOTH SIDES

Why? Take 5 from both sides: 7 − 5 = 2.

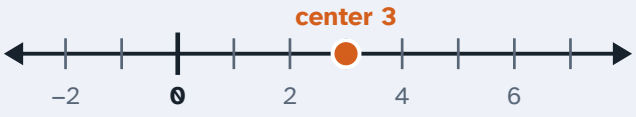


Step 3 of 6

The distance from x to 3 is at most 2

ABSOLUTE VALUE = DISTANCE

Why? Every answer is a point whose distance from 3 is at most 2.

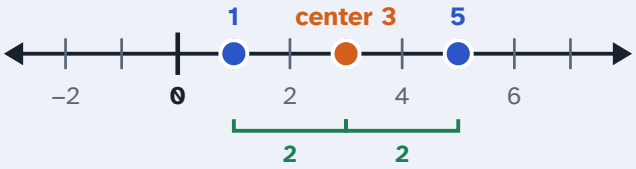


Step 4 of 6

$3 - 2 = 1, \quad 3 + 2 = 5$

GO R EACH WAY FROM THE CENTER

Why? The points exactly 2 away are 2 to the left and 2 to the right of 3. These are the boundary points.

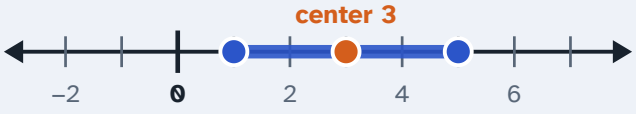


Step 5 of 6

$1 \le x \le 5$

LESS THAN: BETWEEN (AND)

Why? Closer than 2 means staying between the boundaries: x is more than 1 **and** less than 5. The ends count, so the circles are filled.



Step 6 of 6

[1, 5] Check $x = 3$: $|3 - 3| = 0 \le 2$ ✓

INTERVAL NOTATION + CHECK

Why? One piece, written [left, right] or (left, right). Testing a point from the answer in the original problem confirms it.



12 Sign charts for polynomial and rational inequalities

GRADES 11–12

A product or quotient can only change sign where one of its factors is zero (or where it's undefined). Mark those points, test one number in each region, and read the answer off the line.

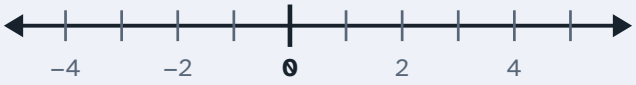
Example 1

Solve: $(x + 2)(x - 3) > 0$

Step 1 of 5

Solve: $(x + 2)(x - 3) > 0$

Why? The sign of $(x + 2)(x - 3)$ can only switch where a factor is zero. Between those points the sign stays the same, so one test number per region is enough.



Step 2 of 5

$x + 2 = 0 \rightarrow x = -2$; $x - 3 = 0 \rightarrow x = 3$

CRITICAL POINTS

Why? These 2 numbers cut the line into 3 regions.



Step 3 of 5

test x	−3	0.5	4
$(x + 2)$	−	+	+
$(x - 3)$	−	−	+
result	+	−	+

TEST A POINT IN EACH REGION

Why? Plug a number from each region into every factor and just track the signs. An even count of minus signs gives +, an odd count gives −. Notice the signs alternate, because each factor flips sign once.

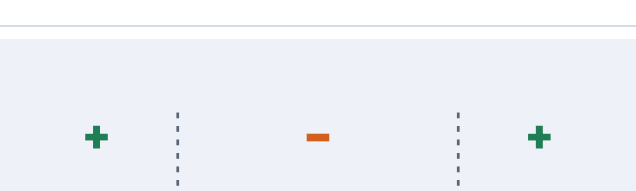


Step 4 of 5

Want > 0 ⇒ keep the + regions

< OR >: LEAVE OUT THE ZEROS

Why? Greater than 0 means positive. The critical points themselves make it 0 (or undefined), not strictly positive or negative, so they're left out (open circles).

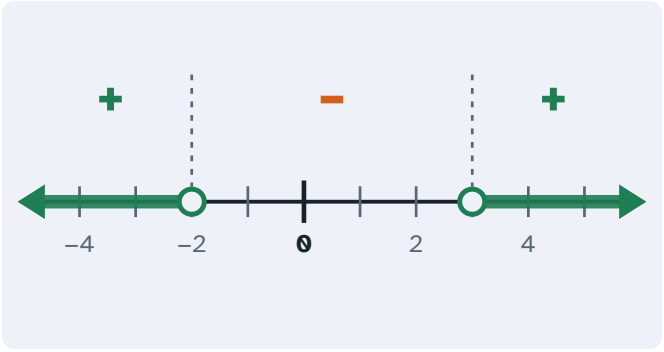


Step 5 of 5

$x \in (-\infty, -2) \cup (3, \infty)$

ANSWER IN INTERVAL NOTATION

Why? Read the shaded pieces from left to right. ∪ joins separate pieces; [ ] means the end is included, ( ) means it isn't. This same chart method works for any polynomial or rational inequality you meet in precalculus and calculus.



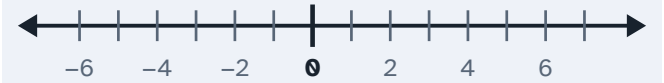
Example 2

Solve: $(x + 4)(x - 1)(x - 5) \le 0$

Step 1 of 5

Solve: $(x + 4)(x - 1)(x - 5) \le 0$

Why? The sign of $(x + 4)(x - 1)(x - 5)$ can only switch where a factor is zero. Between those points the sign stays the same, so one test number per region is enough.

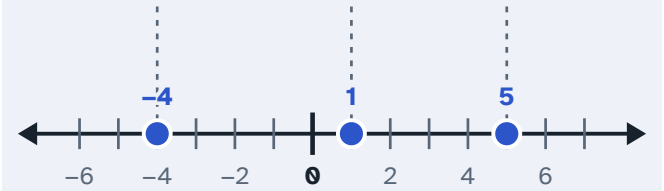


Step 2 of 5

$x + 4 = 0 \to x = -4$; $x - 1 = 0 \to x = 1$; $x - 5 = 0 \to x = 5$

CRITICAL POINTS

Why? These 3 numbers cut the line into 4 regions.

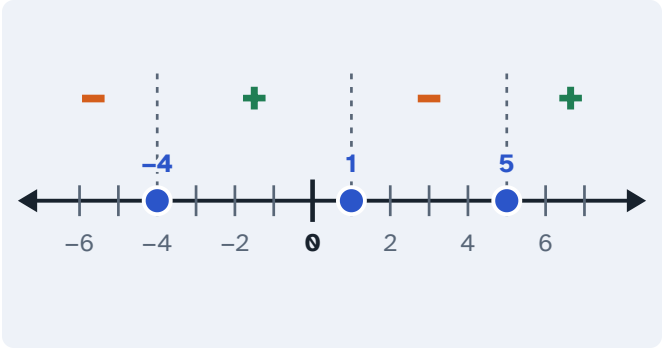


Step 3 of 5

test x	−5	−1.5	3	6
$(x + 4)$	−	+	+	+
$(x - 1)$	−	−	+	+
$(x - 5)$	−	−	−	+
result	−	+	−	+

TEST A POINT IN EACH REGION

Why? Plug a number from each region into every factor and just track the signs. An even count of minus signs gives +, an odd count gives −. Notice the signs alternate, because each factor flips sign once.

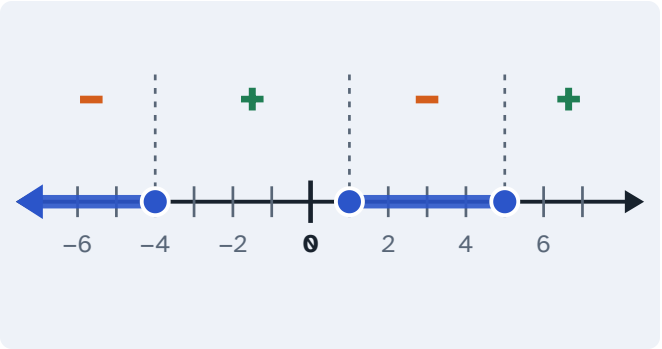


Step 4 of 5

Want ≤ 0 ⇒ keep the – regions

≤ OR ≥: INCLUDE THE ZEROS

Why? Less than 0 means negative. Because it's "or equal to", the points where the top is zero count too (filled circles).

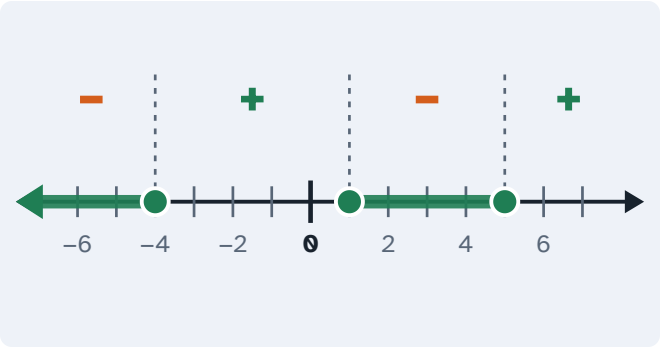


Step 5 of 5

$x \in (-\infty, -4] \cup [1, 5]$

ANSWER IN INTERVAL NOTATION

Why? Read the shaded pieces from left to right. ∪ joins separate pieces; [ ] means the end is included, ( ) means it isn't. This same chart method works for any polynomial or rational inequality you meet in precalculus and calculus.



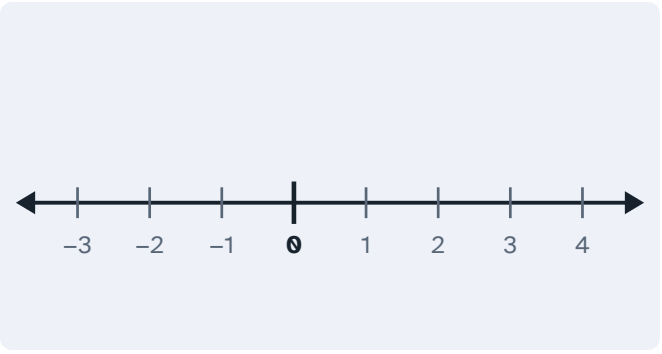
Example 3

Solve: $\frac{x-2}{x+1} \geq 0$

Step 1 of 5

Solve: $\frac{x-2}{x+1} \geq 0$

Why? The sign of $\frac{x-2}{x+1}$ can only switch where a factor is zero or where the bottom is zero (undefined). Between those points the sign stays the same, so one test number per region is enough.

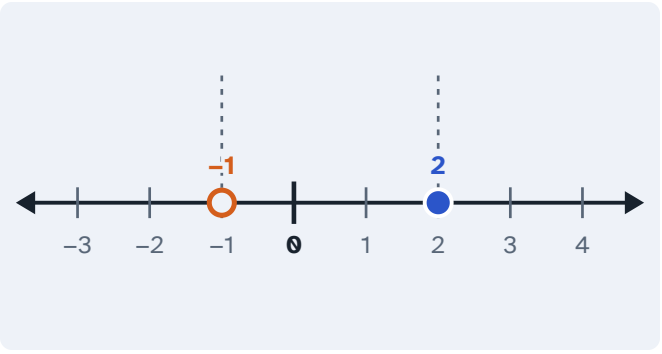


Step 2 of 5

$x - 2 = 0 \rightarrow x = 2$; $x + 1 = 0 \rightarrow x = -1$

CRITICAL POINTS

Why? These 2 numbers cut the line into 3 regions. $x = -1$ makes the bottom zero, so the expression is undefined there: it can never be part of the answer (open circle, orange).

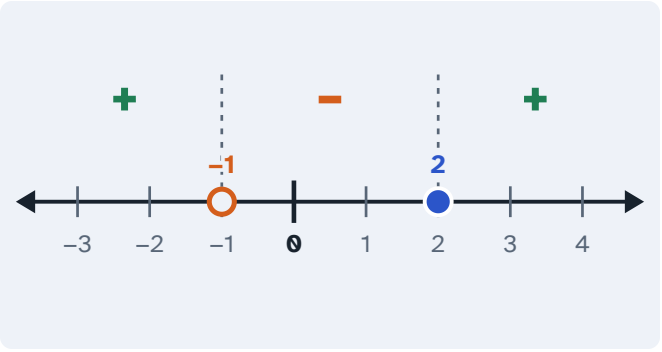


Step 3 of 5

test x	-2	0.5	3
$(x - 2)$	−	−	+
$(x + 1)$	−	+	+
result	+	−	+

TEST A POINT IN EACH REGION

Why? Plug a number from each region into every factor and just track the signs. An even count of minus signs gives +, an odd count gives −. Notice the signs alternate, because each factor flips sign once.

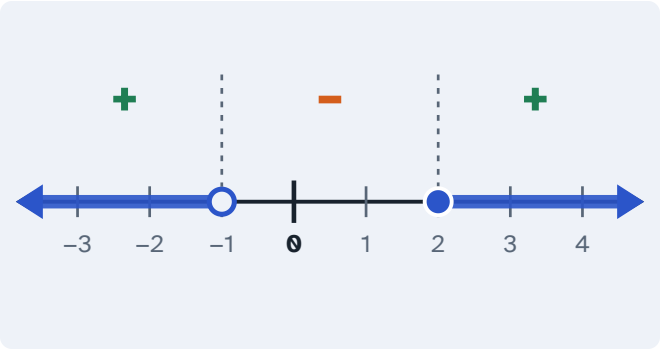


Step 4 of 5

Want ≥ 0 ⇒ keep the + regions

≤ OR ≥: INCLUDE THE ZEROS

Why? Greater than 0 means positive. Because it's "or equal to", the points where the top is zero count too (filled circles), but −1 never does: you can't divide by zero.

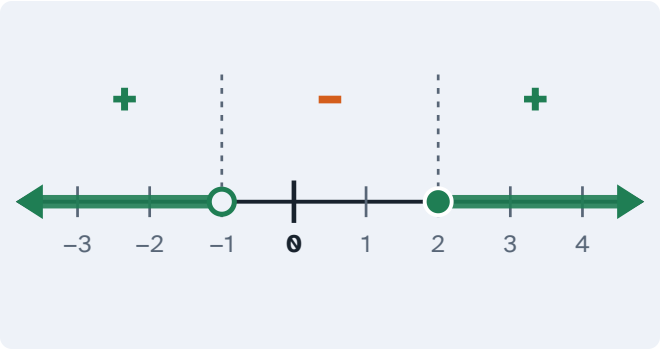


Step 5 of 5

$x \in (-\infty, -1) \cup [2, \infty)$

ANSWER IN INTERVAL NOTATION

Why? Read the shaded pieces from left to right. ∪ joins separate pieces; [ ] means the end is included, ( ) means it isn't. This same chart method works for any polynomial or rational inequality you meet in precalculus and calculus.



Right is bigger. Adding a negative moves left. Subtracting is adding the opposite. Distance is never negative.